Lecture 10: Fast Reinforcement Learning [1]

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CS234 Reinforcement Learning

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[1]With many slides from or derived from David Silver, Examples new

Refresh Your Knowledge. Policy Gradient

- Policy gradient algorithms change the policy parameters using gradient descent on the mean squared Bellman error
  1. True
  2. False.
  3. Not sure
- We can use the performance difference lemma / relative policy performance to: (Select all that are true)
  1. [T] Bound the difference in value between two policies using the advantage function of one policy, and samples from the other policy
  2. [T] Approximately bound the difference in value between two policies using the advantage function of policy 1, importance weights between the two policies, and samples from policy 1
  3. [F] The approximation error in the relative policy performance bounds is bounded by the KL divergence between the states visited under one policy, vs the other [T] $E_s KL(\pi_1, \pi_2)$
  4. These ideas are used in PPO
  5. Not sure

Refresh Your Knowledge. Policy Gradient Answers

- Policy gradient algorithms change the policy parameters using gradient descent on the mean squared Bellman error
  1. True
  2. False.
- We can use the performance difference lemma / relative policy performance to: (Select all that are true)
  1. Bound the difference in value between two policies using the advantage function of one policy, and samples from the other policy
  2. Approximately bound the difference in value between two policies using the advantage function of policy 1, importance weights between the two policies, and samples from policy 1
  3. The approximation error in the relative policy performance bounds is bounded by the KL divergence between the states visited under one policy, vs the other
  4. These ideas are used in PPO

Answer: Policy gradient algorithms generally do gradient ascent on the value function. For the second question, 1, 2 and 4 are true. The approximation error is bounded by the average (over the states visited by one policy) of KL divergence between the two policies.

Class Structure

- Last time: Policy Gradient
- **This time: Fast Learning** sample efficient learning
- Next time: Fast Learning

Up Till Now

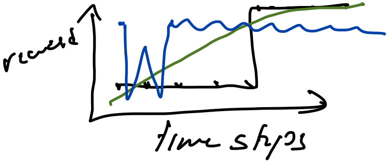
- Discussed optimization, generalization, delayed consequences

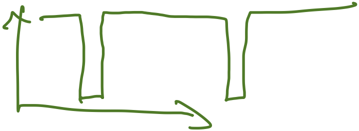
Computational Efficiency and Sample Efficiency

Computational Efficiency	Sample Efficiency
Atari robot simulator	mobile health ed tech software consumer marketing

Evaluation Criteria

- How do we evaluate how "good" an algorithm is?
- If converges? GLIE
- If converges to optimal policy?
- How quickly reaches optimal policy?
- Mistakes make along the way?
- Will introduce different measures to evaluate RL algorithms





Settings, Frameworks & Approaches

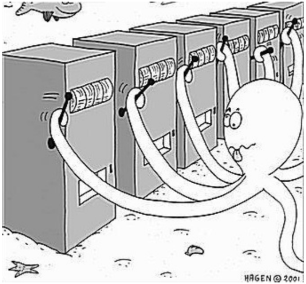
- Over next couple lectures will consider 2 settings, multiple frameworks, and approaches
- Settings: Bandits (single decisions), MDPs
- Frameworks: evaluation criteria for formally assessing the quality of a RL algorithm
- Approaches: Classes of algorithms for achieving particular evaluation criteria in a certain set
- Note: We will see that some approaches can achieve multiple frameworks in multiple settings

Today

- Setting: Introduction to multi-armed bandits & Approach: greedy methods
- Framework: Regret
- Approach: ϵ-greedy methods
- Approach: Optimism under uncertainty
- Framework: Bayesian regret
- Approach: Probability matching / Thompson sampling

Multiarmed Bandits

- Multi-armed bandit is a tuple of $(\mathcal{A}, \mathcal{R})$
- $\mathcal{A}$: known set of m actions (arms)
- $\mathcal{R}^a(r) = \mathbb{P}[r \mid a]$ is an unknown probability distribution over rewards
- At each step t the agent selects an action $a_t \in \mathcal{A}$
- The environment generates a reward $r_t \sim \mathcal{R}^{a_t}$
- Goal: Maximize cumulative reward $\sum_{\tau=1}^{t} r_\tau$



Toy Example: Ways to Treat Broken Toes[1]

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 possible options: (1) surgery (2) buddy taping the broken toe with another toe, (3) do nothing
- Outcome measure / reward is binary variable: whether the toe has healed (+1) or not healed (0) after 6 weeks, as assessed by x-ray

[1] Note: This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Check Your Understanding: Bandit Toes [1]

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 common options: (1) surgery (2) buddy taping the broken toe with another toe (3) doing nothing
- Outcome measure is binary variable: whether the toe has healed (+1) or not (0) after 6 weeks, as assessed by x-ray
- Model as a multi-armed bandit with 3 arms, where each arm is a Bernoulli variable with an unknown parameter θ_i
- Select all that are true ↙ which treatment give
  1. F — Pulling an arm / taking an action corresponds to whether the toe has healed or not
  2. F — A multi-armed bandit is a better fit to this problem than a MDP because treating each patient involves ~~multiple~~ decisions single decision
  3. T — After treating a patient, if $\theta_i \neq 0$ and $\theta_i \neq 1$ $\forall i$ sometimes a patient's toe will heal and sometimes it may not
  4. Not sure

[1] Note: This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Check Your Understanding: Bandit Toes Solution [1]

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 common options: (1) surgery (2) buddy taping the broken toe with another toe (3) doing nothing
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- Model as a multi-armed bandit with 3 arms, where each arm is a Bernoulli variable with an unknown parameter θ_i
- Select all that are true
  1. Pulling an arm / taking an action corresponds to whether the toe has healed or not
  2. A multi-armed bandit is a better fit to this problem than a MDP because treating each patient involves multiple decisions
  3. After treating a patient, if $\theta_i \neq 0$ and $\theta_i \neq 1$ $\forall i$ sometimes a patient's toe will heal and sometimes it may not
  4. Not sure

  3 is true. Pulling an arm corresponds to treating a patient. A MAB is a better fit than a MDP, because actions correspond to treating a patient, and the treatment of one patient does not influence that next patient that comes to be treated.

[1]

Greedy Algorithm

- We consider algorithms that estimate $\hat{Q}_t(a) \approx Q(a) = \mathbb{E}[R(a)]$
- Estimate the value of each action by Monte-Carlo evaluation

$$\hat{Q}_t(a) = \frac{1}{N_t(a)} \sum_{i=1}^{t-1} r_i \mathbb{1}(a_i = a)$$

- The **greedy** algorithm selects the action with highest value

$$a_t^* = \arg\max_{a \in \mathcal{A}} \hat{Q}_t(a)$$

Toy Example: Ways to Treat Broken Toes[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$

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Toy Example: Ways to Treat Broken Toes, Greedy[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
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- Greedy
  1. Sample each arm once
     - Take action a^1 ($r \sim$Bernoulli(0.95)), get 0, $\hat{Q}(a^1) = 0$
     - Take action a^2 ($r \sim$Bernoulli(0.90)), get $+1$, $\hat{Q}(a^2) = 1$
     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. What is the probability of greedy selecting each arm next? Assume ties are split uniformly.

could have gotten 1

$p(a_2) = 1$

[1] Note: This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes, Greedy[2]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
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  1. Sample each arm once
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     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Will the greedy algorithm ever find the best arm in this case? no

[2]Note:This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Greedy Algorithm

- We consider algorithms that estimate $\hat{Q}_t(a) \approx Q(a) = \mathbb{E}[R(a)]$
- Estimate the value of each action by Monte-Carlo evaluation

$$\hat{Q}_t(a) = \frac{1}{N_t(a)} \sum_{t=1}^{T} r_t \mathbb{1}(a_t = a)$$

- The **greedy** algorithm selects the action with highest value

$$a_t^* = \arg\max_{a \in \mathcal{A}} \hat{Q}_t(a)$$

- **Greedy can lock onto suboptimal action, forever**

Today

- Setting: Introduction to multi-armed bandits & Approach: greedy methods
- **Framework: Regret**
- Approach: ϵ-greedy methods
- Approach: Optimism under uncertainty
- Framework: Bayesian regret
- Approach: Probability matching / Thompson sampling

Assessing the Performance of Algorithms

- How do we evaluate the quality of a RL (or bandit) algorithm?
- So far: computational complexity, convergence, convergence to a fixed point, & empirical performance performance
- Today: introduce a formal measure of how well a RL/bandit algorithm will do in any environment, compared to optimal

Regret

- **Action-value** is the mean reward for action a

$$Q(a) = \mathbb{E}[r \mid a]$$

- **Optimal value** V^*

$$V^* = Q(a^*) = \max_{a \in \mathcal{A}} Q(a)$$

- **Regret** is the opportunity loss for one step

$$l_t = \mathbb{E}[V^* - Q(a_t)]$$

arm/action selected at the step t

Regret

- **Action-value** is the mean reward for action a

$$Q(a) = \mathbb{E}[r \mid a]$$

- **Optimal value** V^*

$$V^* = Q(a^*) = \max_{a \in \mathcal{A}} Q(a)$$

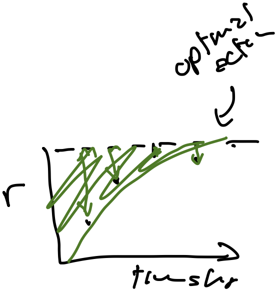
- **Regret** is the opportunity loss for one step

$$l_t = \mathbb{E}[V^* - Q(a_t)]$$

- **Total Regret** is the total opportunity loss

$$L_t = \mathbb{E}[\sum_{\tau=1}^{t} V^* - Q(a_\tau)]$$

- Maximize cumulative reward $\Longleftrightarrow$ minimize total regret



Evaluating Regret

- **Count** $N_t(a)$ is number of times action a has been selected
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$
- Regret is a function of gaps and counts

expectation

$$
\begin{aligned}
L_t &= \mathbb{E}\left[\sum_{\tau=1}^{t} V^* - Q(a_\tau)\right] \\
&= \sum_{a\in\mathcal{A}} \mathbb{E}[N_t(a)](V^* - Q(a)) \\
&= \sum_{a\in\mathcal{A}} \mathbb{E}[N_t(a)]\Delta_a
\end{aligned}
$$

- A good algorithm ensures small counts for large gap,s but gaps are not known

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

- True (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
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  - doing nothing: $Q(a^3) = \theta_3 = .1$
- Greedy

Action	Optimal Action	Observed Reward	Regret
a^1	a^1	0	0
a^2	a^1	1	.95-.9= .05
a^3	a^1	0	.95-.1=.85
a^2	a^1	1	
a^2	a^1	0	

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

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- Greedy

Action	Optimal Action	Observed Reward	Regret
a^1	a^1	0	0
a^2	a^1	1	0.05
a^3	a^1	0	0.85
a^2	a^1	1	0.05
a^2	a^1	0	0.05

- Regret for greedy methods can be **linear** in the number of decisions made (timestep)

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

- Greedy

Action	Optimal Action	Observed Reward	Regret
a^1	a^1	0	0
a^2	a^1	1	0.05
a^3	a^1	0	0.85
a^2	a^1	1	0.05
a^2	a^1	0	0.05

- **Note: in real settings we cannot evaluate the regret because it requires knowledge of the expected reward of the true best action.**
- Instead we can prove an upper bound on the potential regret of an algorithm in **any bandit** problem

Today

- Setting: Introduction to multi-armed bandits & Approach: greedy methods
- Framework: Regret
- **Approach: ϵ-greedy methods**
- Approach: Optimism under uncertainty
- Framework: Bayesian regret
- Approach: Probability matching / Thompson sampling

ϵ-Greedy Algorithm

- The ϵ-**greedy** algorithm proceeds as follows:
  - With probability $1 - \epsilon$ select $a_t = \arg\max_{a \in \mathcal{A}} \hat{Q}_t(a)$
  - With probability ϵ select a random action
- Always will be making a sub-optimal decision ϵ fraction of the time
- Already used this in prior homeworks

Toy Example: Ways to Treat Broken Toes, ϵ-Greedy[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$
- ϵ-greedy
  1. Sample each arm once
     - Take action a^1 (r ~Bernoulli(0.95)), get $+1$, $\hat{Q}(a^1) = 1$
     - Take action a^2 (r ~Bernoulli(0.90)), get $+1$, $\hat{Q}(a^2) = 1$
     - Take action a^3 (r ~Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Let $\epsilon = 0.1$
  3. What is the probability ϵ-greedy will pull each arm next? Assume ties are split uniformly.

90% greedy: 45% a_1 45% a_2
10% .033 a_1 .033 a_2 .033 a_3

[1]Note: This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

- True (unknown) Bernoulli reward parameters for each arm (action) are
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- UCB1 (Auer, Cesa-Bianchi, Fischer 2002)

Action	Optimal Action	Regret
a^1	a^1	
a^2	a^1	
a^3	a^1	
a^1	a^1	
a^2	a^1	

$\frac{T\epsilon}{|A|}$

T decisions

- Will ϵ-greedy ever select a^3 again? If ϵ is fixed, how many (minimum) times will each arm be selected?

Recall: Bandit Regret

- **Count** $N_t(a)$ is expected number of selections for action a
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$
- Regret is a function of gaps and counts

$$L_t = \mathbb{E}\left[\sum_{\tau=1}^{t} V^* - Q(a_\tau)\right]$$

$$= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)](V^* - Q(a))$$

$$= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)]\Delta_a$$

$$\sum_a \frac{\epsilon}{|A|} T \Delta_a + \ldots$$

- A good algorithm ensures small counts for large gap, but gaps are not known

Check Your Understanding: ϵ-greedy Bandit Regret

- **Count** $N_t(a)$ is expected number of selections for action a
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$
- Regret is a function of gaps and counts

$$L_t = \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)] \Delta_a$$

- Informally an algorithm has linear regret if it takes a non-optimal action a constant fraction of the time
- Assume $\exists a \; s.t. \; \Delta_a > 0$
- Select all
  1. $\epsilon = 0.1$ ϵ-greedy can have linear regret
  2. $\epsilon = 0$ ϵ-greedy can have linear regret
  3. Not sure

Check Your Understanding: ϵ-greedy Bandit Regret Answer

- **Count** $N_t(a)$ is expected number of selections for action a
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$
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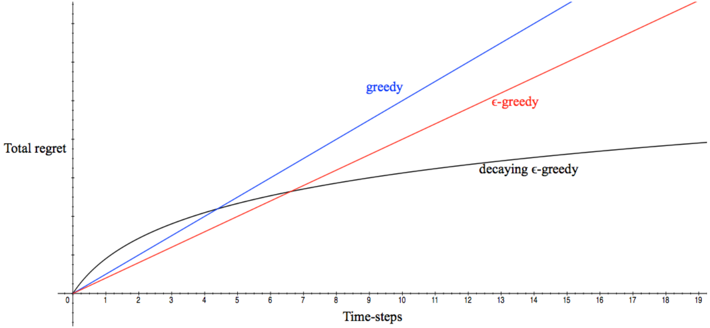
- Informally an algorithm has linear regret if it takes a non-optimal action a constant fraction of the time
- Assume $\exists a \; s.t. \; \Delta_a > 0$
- Select all
  1. $\epsilon = 0.1$ ϵ-greedy can have linear regret
  2. $\epsilon = 0$ ϵ-greedy can have linear regret
  3. Not sure

Both can have linear regret.

"Good": Sublinear or below regret



- **Explore forever**: have linear total regret
- **Explore never**: have linear total regret
- Is it possible to achieve sublinear (in the time steps/number of decisions made) regret?

Types of Regret bounds

- **Problem independent**: Bound how regret grows as a function of T, the total number of time steps the algorithm operates for
- **Problem dependent**: Bound regret as a function of the number of times we pull each arm and the gap between the reward for the pulled arm and a^*

Lower Bound

- Use lower bound to determine how hard this problem is
- The performance of any algorithm is determined by similarity between optimal arm and other arms
- Hard problems have similar looking arms with different means
- This is described formally by the gap Δ_a and the similarity in distributions $D_{KL}(\mathcal{R}^a \| \mathcal{R}^{a^*})$
- Theorem (Lai and Robbins): Asymptotic total regret is at least logarithmic in number of steps

$$\lim_{t \to \infty} L_t \geq \log t \sum_{a | \Delta_a > 0} \frac{\Delta_a}{D_{KL}(\mathcal{R}^a \| \mathcal{R}^{a^*})}$$

- Promising in that lower bound is sublinear

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- Framework: Regret
- Approach: ϵ-greedy methods
- **Approach: Optimism under uncertainty**
- Framework: Bayesian regret
- Approach: Probability matching / Thompson sampling

Approach: Optimism in the Face of Uncertainty

- Choose actions that that might have a high value
- Why?
- Two outcomes:
  - get high reward
  - learn something

Approach: Optimism in the Face of Uncertainty

- Choose actions that that might have a high value
- Why?
- Two outcomes:
  - Getting high reward: if the arm really has a high mean reward
  - Learn something: if the arm really has a lower mean reward, pulling it will (in expectation) reduce its average reward and the uncertainty over its value

Upper Confidence Bounds

true unknown mean reward for action a

- Estimate an upper confidence $U_t(a)$ for each action value, such that $Q(a) \leq U_t(a)$ with high probability
- This depends on the number of times $N_t(a)$ action a has been selected
- Select action maximizing Upper Confidence Bound (UCB)

$$a_t = \arg\max_{a \in \mathcal{A}} [U_t(a)]$$

Hoeffding's Inequality

- Theorem (Hoeffding's Inequality): Let $X_1, \dots, X_n$ be i.i.d. random variables in $[0, 1]$, and let $\bar{X}_n = \frac{1}{n}\sum_{\tau=1}^{n} X_\tau$ be the sample mean. Then

$$\mathbb{P}\left[\mathbb{E}[X] > \bar{X}_n + u\right] \leq \exp(-2nu^2)$$

$$P(|E[X] - \bar{X}_n| > u) \leq 2\exp(-2nu^2) \doteq \delta$$

$$\exp(-2nu^2) = \delta/2$$

$$u^2 = 1/n \log 2/\delta$$

$$u = \sqrt{\tfrac{1}{n}\log 2/\delta}$$

$$\bar{X}_n - u \leq E[X] \leq \bar{X}_n + u$$

with prob $\geq 1-\delta$

UCB Bandit Regret

- This leads to the UCB1 algorithm

empirical avg

$$a_t = \arg\max_{a \in \mathcal{A}} \left[\hat{Q}(a) + \sqrt{\frac{2 \log \frac{1}{\delta}}{N_t(a)}} \right]$$

\# samples of a after t time steps

Toy Example: Ways to Treat Broken Toes, Thompson Sampling[1]

- True (unknown) parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$
- Optimism under uncertainty, UCB1 (Auer, Cesa-Bianchi, Fischer 2002)
  1. Sample each arm once

[1] Note:This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes, Optimism[1]

- True (unknown) parameters for each arm (action) are
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Toy Example: Ways to Treat Broken Toes, Optimism[1]

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     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log \frac{1}{\delta}}{N_t(a)}}$$

a_1: $1 + \sqrt{\frac{2 \log 1/\delta}{1}}$ a_2: $1 + \sqrt{\frac{2 \log 1/\delta}{1}}$

$a_3 = 0 + \sqrt{\frac{2 \log 1/\delta}{1}}$

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  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log \frac{1}{\delta}}{N_t(a)}}$$

  3. $t = 3$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

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- UCB1 (Auer, Cesa-Bianchi, Fischer 2002)
  1. Sample each arm once
     - Take action a^1 ($r \sim$Bernoulli(0.95)), get $+1$, $\hat{Q}(a^1) = 1$
     - Take action a^2 ($r \sim$Bernoulli(0.90)), get $+1$, $\hat{Q}(a^2) = 1$
     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log \frac{1}{\delta}}{N_t(a)}}$$

  3. $t = t + 1$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

[1] Note:This is a made up example. This is not the actual expected efficacies of the

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret

- True (unknown) parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$
- UCB1 (Auer, Cesa-Bianchi, Fischer 2002)

Action	Optimal Action	Regret
a^1	a^1	
a^2	a^1	
a^3	a^1	
a^1	a^1	
a^2	a^1	

Confidence Level δ

- Subtle
- If there are a fixed number of time steps T for the problem setting, can set $\delta = \frac{\delta}{T}$ |A|
  - Union bound: $P(\cup E_i) \leq \sum_i P(E_i)$
- Often want to do this in other settings

High Probability Regret Bound for UCB Multi-armed Bandit

- Any sub-optimal arm $a \neq a^*$ is pulled by UCB at most $\mathbb{E}N_T(a) \leq C' \frac{\log \frac{1}{\delta}}{\Delta_a^2} + \frac{\pi^2}{3} + 1$.
  So the regret of UCB is bounded by $\sum_a \Delta_a \mathbb{E}N_T(a) \leq \sum_a C' \frac{\log T}{\Delta_a} + |A|(\frac{\pi^2}{3} + 1)$.
  (Arm means $\in [0, 1]$)

under event that confid intervals hold

true, empirical

$1/\delta$ but think of as T

$$P\left(|Q(a) - \hat{Q}_t(a)| \geq \sqrt{\frac{C \log \frac{1}{\delta}}{N_t(a)}}\right) \leq \frac{\delta}{T} \quad (1)$$

a suboptimal $a \neq a^*$ $\Delta a \neq 0$ action a

$Q(a) - \sqrt{\frac{C \log 1/\delta}{N_t(a)}} \leq \hat{Q}_t(a) \leq Q(a) + \sqrt{\frac{C \log 1/\delta}{N_t(a)}}$ if CI hold

UCB

$\hat{Q}_t(a) + \sqrt{\frac{C \log 1/\delta}{N_t(a)}} \geq \hat{Q}_t(a^*) + \sqrt{\frac{C \log 1/\delta}{N_t(a^*)}} \geq Q(a^*)$ if CI hold

$Q(a) + \sqrt{\frac{C \log 1/\delta}{N_t(a)}} \cdot 2 \geq Q(a^*)$

$2\sqrt{\frac{C \log 1/\delta}{N_t(a)}} \geq Q(a^*) - Q(a) = \Delta a$

High Probability Regret Bound for UCB Multi-armed Bandit

- Any sub-optimal arm $a \neq a^*$ is pulled by UCB at most $\mathbb{E}N_T(a) \leq C' \frac{\log \frac{1}{\delta}}{\Delta_a^2} + \frac{\pi^2}{3} + 1$. So the regret of UCB is bounded by $\sum_a \Delta_a \mathbb{E}N_T(a) \leq \sum_a C' \frac{\log T}{\Delta_a} + |A|(\frac{\pi^2}{3} + 1)$. (Arm means $\in [0,1]$)

$$Q(a) - \sqrt{\frac{C \log \frac{1}{\delta}}{N_t(a)}} \leq \hat{Q}_t(a) \leq Q(a) + \sqrt{\frac{C \log \frac{1}{\delta}}{N_t(a)}} \tag{2}$$

$$\hat{Q}_t(a) + \sqrt{\frac{C \log \frac{1}{\delta}}{N_t(a)}} \geq \hat{Q}_t(a^*) + \sqrt{\frac{C \log \frac{1}{\delta}}{N_t(a^*)}} \geq Q(a^*) \tag{3}$$

$$\Rightarrow \quad Q(a) + 2\sqrt{\frac{C \log \frac{1}{\delta}}{N_t(a)}} \geq Q(a^*) \tag{4}$$

$$2\sqrt{\frac{C \log \frac{1}{\delta}}{N_t(a)}} \geq Q(a^*) - Q(a) = \Delta_a \tag{5}$$

$\log T/\delta$

$$N_t(a) \leq \frac{4C \log \frac{1}{\delta}}{\Delta_a^2} \tag{6}$$

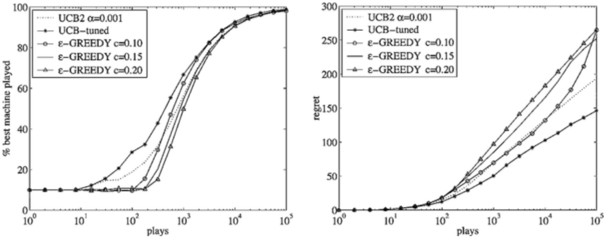
UCB Bandit Regret

- This leads to the UCB1 algorithm

$$a_t = \arg\max_{a \in \mathcal{A}} \left[\hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}} \right]$$

- Theorem: The UCB algorithm achieves logarithmic asymptotic total regret

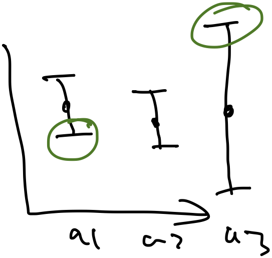
$$\lim_{t \to \infty} L_t \leq 8 \log t \sum_{a | \Delta_a > 0} \frac{1}{\Delta_a}$$



Optional Check Your Understanding

to be answered Thurs

- An alternative would be to always select the arm with the highest lower bound
- Why can this yield linear regret?
- Consider a two arm case for simplicity



Today

- Setting: Introduction to multi-armed bandits & Approach: greedy methods
- Framework: Regret
- Approach: ϵ-greedy methods
- Approach: Optimism under uncertainty
- Note: bandits are a simpler place to see these ideas, but these ideas will extend to MDPs
- Next time: more fast learning