Lecture 12: Fast RL Part III[1]

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CS234 Reinforcement Learning

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[1]With a few slides derived from David Silver

Refresh Your Knowledge Fast RL Part II

- The prior over arm 1 is Beta(1,2) (left) and arm 2 is a Beta(1,1) (right figure). Select all that are true.
  1. Sample 3 params: 0.1,0.5,0.3. These are more likely to come from the Beta(1,2) distribution than Beta(1,1).
  2. Sample 3 params: 0.2,0.5,0.8. These are more likely to come from the Beta(1,1) distribution than Beta(1,2).
  3. It is impossible that the true Bernoulli parameter is 0 if the prior is Beta(1,1).
  4. Not sure
- The prior over arm 1 is Beta(1,2) (left) and arm 2 is a Beta(1,1) (right). The true parameters are arm 1 $\theta_1 = 0.4$ & arm 2 $\theta_2 = 0.6$. Thompson sampling = TS
  1. TS could sample $\theta = 0.5$ (arm 1) and $\theta = 0.55$ (arm 2).
  2. For the sampled thetas (0.5,0.55) TS is optimistic with respect to the true arm parameters for all arms.
  3. For the sampled thetas (0.5,0.55) TS will choose the true optimal arm for this round.
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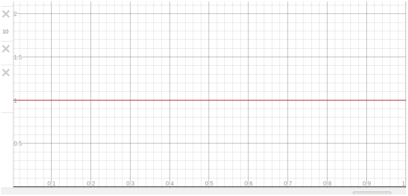

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Class Structure

- Last time: Fast Learning (Bayesian bandits to MDPs)
- **This time: Fast Learning III (MDPs)**
- Next time: Batch RL

Settings, Frameworks & Approaches

- Over these 3 lectures will consider 2 settings, multiple frameworks, and approaches
- Settings: Bandits (single decisions), MDPs
- Frameworks: evaluation criteria for formally assessing the quality of a RL algorithm. So far seen empirical evaluations, asymptotic convergence, regret, probably approximately correct
- Approaches: Classes of algorithms for achieving particular evaluation criteria in a certain set. So far for exploration seen: greedy, $\epsilon-$greedy, optimism, Thompson sampling, for multi-armed bandits
- **Goal: fast, efficient RL for large, complex domains.**

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Fast RL in Markov Decision Processes

- Very similar set of frameworks and approaches are relevant for fast learning in reinforcement learning
- Frameworks
  - Regret
  - Bayesian regret
  - Probably approximately correct (PAC)
- Approaches
  - Optimism under uncertainty
  - Probability matching / Thompson sampling
- Framework: Probably approximately correct

Model-Based Interval Estimation with Exploration Bonus (MBIE-EB)

(Strehl and Littman, J of Computer & Sciences 2008)

Given ϵ, δ, m

$\beta = \frac{1}{1-\gamma}\sqrt{0.5 \ln(2|S||A|m/\delta)}$

$n_{sas}(s, a, s') = 0$, $\forall s \in S$, $a \in A$, $s' \in S$

$rc(s, a) = 0$, $n_{sa}(s, a) = 0$, $\tilde{Q}(s, a) = 1/(1-\gamma)$, $\forall\ s \in S$, $a \in A$

$t = 0$, $s_t = s_{init}$

loop

  $a_t = \arg\max_{a \in \mathcal{A}} \tilde{Q}(s_t, a)$

  Observe reward r_t and state s_{t+1}

  $n_{sa}(s_t, a_t) = n_{sa}(s_t, a_t) + 1$, $n_{sas}(s_t, a_t, s_{t+1}) = n_{sas}(s_t, a_t, s_{t+1}) + 1$

  $rc(s_t, a_t) = \frac{rc(s_t, a_t)(n_{sa}(s_t, a_t) - 1) + r_t}{n_{sa}(s_t, a_t)}$

  $\hat{R}(s_t, a_t) = rc(s_t, a_t)$ and $\hat{T}(s'|s_t, a_t) = \frac{n_{sas}(s_t, a_t, s')}{n_{sa}(s_t, a_t)}$, $\forall s' \in S$

  while not converged **do**

    $\tilde{Q}(s, a) = \hat{R}(s, a) + \gamma \sum_{s'} \hat{T}(s'|s, a) \max_{a'} \tilde{Q}(s', a) + \frac{\beta}{\sqrt{n_{sa}(s, a)}}$, $\forall\ s \in S$, $a \in A$

  end while

end loop

Framework: PAC for MDPs

- For a given ϵ and δ, A RL algorithm $\mathcal{A}$ is PAC if on all but N steps, the action selected by algorithm $\mathcal{A}$ on time step t, a_t, is ϵ-close to the optimal action, where N is a polynomial function of $(|S|, |A|, \frac{1}{1-\gamma}, \frac{1}{\epsilon}, \frac{1}{\delta})$
- Is this true for all algorithms?

MBIE-EB is a PAC RL Algorithm

Theorem 2. *Suppose that ϵ and δ are two real numbers between 0 and 1 and $M = \langle S, A, T, \mathcal{R}, \gamma \rangle$ is any MDP. There exists an input $m = m(\frac{1}{\epsilon}, \frac{1}{\delta})$, satisfying $m(\frac{1}{\epsilon}, \frac{1}{\delta}) = O(\frac{|S|}{\epsilon^2(1-\gamma)^4} + \frac{1}{\epsilon^2(1-\gamma)^4} \ln \frac{|S||A|}{\epsilon(1-\gamma)\delta})$, and $\beta = (1/(1-\gamma))\sqrt{\ln(2|S||A|m/\delta)/2}$ such that if MBIE-EB is executed on MDP M, then the following holds. Let $\mathcal{A}_t$ denote MBIE-EB's policy at time t and s_t denote the state at time t. With probability at least $1-\delta$, $V_M^{\mathcal{A}_t}(s_t) \geqslant V_M^*(s_t) - \epsilon$ is true for all but $O(\frac{|S||A|}{\epsilon^3(1-\gamma)^6}(|S| + \ln \frac{|S||A|}{\epsilon(1-\gamma)\delta}) \ln \frac{1}{\delta} \ln \frac{1}{\epsilon(1-\gamma)})$ timesteps t.*

A Sufficient Set of Conditions to Make a RL Algorithm PAC

- Strehl, A. L., Li, L., & Littman, M. L. (2006). Incremental model-based learners with formal learning-time guarantees. In Proceedings of the Twenty-Second Conference on Uncertainty in Artificial Intelligence (pp. 485-493)

A Sufficient Set of Conditions to Make a RL Algorithm PAC

1. Optimism
2. Accuracy
3. Bounded learning complexity: number of updates of the state-action Q values, and number of times visit a (s,a) pair for which don't have an accurate estimate of its reward and/or dynamics model.

- Note: the above assumed a tabular domain (finite state and action space). But these ideas relate back to the ideas we saw in UCB, and also are relevant later for function approximation.

One of the key ideas: Simulation Lemma[1]

- Bound error in value function due to error in dynamics & reward models

[1] Covered in problem sessions: https://web.stanford.edu/class/cs234/sessions/CS234_Win23_ProblemSession2.pdf [solutions: https://web.stanford.edu/class/cs234/sessions/CS234_Win23_ProblemSession2_Solutions.pdf] .

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Refresher: Bayesian Bandits

- **Bayesian bandits** exploit prior knowledge of rewards, $p[\mathcal{R}]$
- They compute posterior distribution of rewards $p[\mathcal{R} \mid h_t]$, where $h_t = (a_1, r_1, \ldots, a_{t-1}, r_{t-1})$
- Use posterior to guide exploration
  - Upper confidence bounds (Bayesian UCB)
  - Probability matching (Thompson Sampling)
- Better performance if prior knowledge is accurate

Refresher: Bernoulli Bandits

- Consider a bandit problem where the reward of an arm is a binary outcome $\{0, 1\}$ sampled from a Bernoulli with parameter θ
  - E.g. Advertisement click through rate, patient treatment succeeds/fails, ...
- The Beta distribution $Beta(\alpha, \beta)$ is conjugate for the Bernoulli distribution

$$p(\theta|\alpha, \beta) = \theta^{\alpha-1}(1-\theta)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

where $\Gamma(x)$ is the Gamma function.

- Assume the prior over θ is a $Beta(\alpha, \beta)$ as above
- Then after observed a reward $r \in \{0, 1\}$ then updated posterior over θ is $Beta(r + \alpha, 1 - r + \beta)$

Thompson Sampling for Bandits

Initialize prior over each arm a, $p(\mathcal{R}_a)$
loop
    For each arm a **sample** a reward distribution $\mathcal{R}_a$ from posterior
    Compute action-value function $Q(a) = \mathbb{E}[\mathcal{R}_a]$
    $a_t = \arg\max_{a \in \mathcal{A}} Q(a)$
    Observe reward r
    Update posterior $p(\mathcal{R}_a | r)$ using Bayes law
end loop

Bayesian Model-Based RL

- Maintain posterior distribution over **MDP** models
- Estimate both transition and rewards, $p[\mathcal{P}, \mathcal{R} \mid h_t]$, where $h_t = (s_1, a_1, r_1, \ldots, s_t)$ is the history
- Use posterior to guide exploration
  - Upper confidence bounds (Bayesian UCB)
  - Probability matching (Thompson sampling)

Thompson Sampling: Model-Based RL

- Thompson sampling implements probability matching

$$\pi(s, a \mid h_t) = \mathbb{P}[Q(s,a) \geq Q(s,a'), \forall a' \neq a \mid h_t]$$
$$= \mathbb{E}_{\mathcal{P},\mathcal{R}|h_t}\left[\mathbb{1}(a = \arg\max_{a \in \mathcal{A}} Q(s,a))\right]$$

- Use Bayes law to compute posterior distribution $p[\mathcal{P}, \mathcal{R} \mid h_t]$
- **Sample** an MDP $\mathcal{P}, \mathcal{R}$ from posterior
- Solve MDP using favorite planning algorithm to get $Q^*(s,a)$
- Select optimal action for sample MDP, $a_t = \arg\max_{a \in \mathcal{A}} Q^*(s_t, a)$

Thompson Sampling for MDPs

Initialize prior over the dynamics and reward models for each (s, a), $p(\mathcal{R}_{as})$, $p(\mathcal{T}(s'|s, a))$
Initialize state s_0
loop
    Sample a MDP $\mathcal{M}$: for each (s, a) pair, sample a dynamics model $\mathcal{T}(s'|s, a)$ and reward model $\mathcal{R}(s, a)$
    Compute $Q^*_{\mathcal{M}}$, optimal value for MDP $\mathcal{M}$
    $a_t = \arg\max_{a \in \mathcal{A}} Q^*_{\mathcal{M}}(s_t, a)$
    Observe reward r_t and next state s_{t+1}
    Update posterior $p(\mathcal{R}_{a_t s_t}|r_t)$, $p(\mathcal{T}(s'|s_t, a_t)|s_{t+1})$ using Bayes rule
    $t = t + 1$
end loop

Check Your Understanding: Fast RL III

- Strategic exploration in MDPs (select all):
  1. Doesn't really matter because the distribution of data is independent of the policy followed
  2. Can involve using optimism with respect to both the possible dynamics and reward models in order to compute an optimistic Q function
  3. Is known as PAC if the number of time steps on which a less than near optimal decision is made is guaranteed to be less than an exponential function of the problem domain parameters (state space cardinality, etc).
  4. Not sure
- In Thompson sampling for tabular MDPs in the shown algorithm:
  1. TS samples the reward model parameters and could use the empirical average for the dynamics model parameters and obtain the same performance
  2. Can perform MDP planning everytime the posterior is updated
  3. Always has the same computational cost each step as Q-learning
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Check Your Understanding: Fast RL III Solutions

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Generalization and Strategic Exploration

- Active area of ongoing research: combine generalization & strategic exploration
- Many approaches are grounded by principles outlined here
  - Optimism under uncertainty
  - Thompson sampling

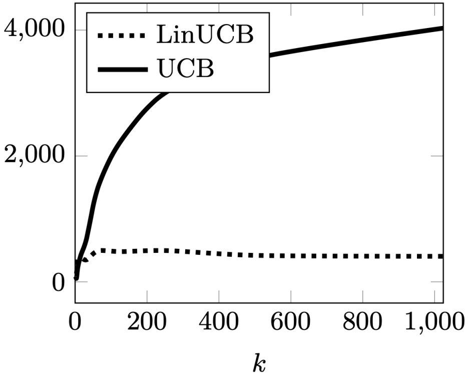
Generalization and Strategic Exploration

- Active area of ongoing research: combine generalization & strategic exploration
- Many approaches are grounded by principles outlined here
  - Optimism under uncertainty
  - Thompson sampling
- These issues are important for large state spaces and large action spaces, in bandits and Markov decision processes
- Rest of today: brief discussion of **contextual bandits**, then MDPs

Contextual Multiarmed Bandits

- Multi-armed bandit is a tuple of $(\mathcal{A}, \mathcal{R})$, where $\mathcal{A}$: known set of m actions (arms)
  - $\mathcal{R}^a(r) = \mathbb{P}[r \mid a]$ is an unknown probability distribution over rewards
  - At each step t the agent selects an action $a_t \in \mathcal{A}$
  - The environment generates a reward $r_t \sim \mathcal{R}^{a_t}$
  - Goal: Maximize cumulative reward $\sum_{\tau=1}^{t} r_\tau$ / minimize total regret
- Contextual bandits: context/state space $\mathcal{S}$ and action space $\mathcal{A}$
  - $\mathcal{R}^{a,s}(r) = \mathbb{P}[r \mid a, s]$ is an unknown probability distribution over rewards, for a particular state and action
  - If the state and/or action space is very large, it is common to use a function to represent the relationship between the input state and action and the output rewards

Benefits of Generalization: Bandits vs Contextual Multiarmed Bandits:



- k is the number of arms, y-axis is the regret. [Figure is Figure 19.1, Lattimore and Szepesvari, Bandit Algorithms]

Contextual Multiarmed Bandits

- Contextual bandits: context/state space $\mathcal{S}$ and action space $\mathcal{A}$
- $\mathcal{R}^{a,s}(r) = \mathbb{P}[r \mid a, s]$ is an unknown probability distribution over rewards, for a particular state and action
- If the state and/or action space is very large, it is common to use a function to represent the relationship between the input state and action and the output rewards
- Common to model reward as a linear function[2] of input features $\phi(s, a)$
- $r = \theta\phi(s, a) + \epsilon$ where $\epsilon \sim$

[2]Notation alert!

Disjoint Linear Contextual Multi-armed Bandits

- Assumes that each arm a has its own θ_a parameter
- $r(s,a) = \theta_a \phi(s) + \epsilon$ where $\epsilon \sim$
- Check your understanding: can $r = \theta\phi(s,a) + \epsilon$ represent a disjoint linear model?

Learning in Linear Contextual Multiarmed Bandits

- $r = \theta\phi(s, a) + \epsilon$
- Previously we used Hoeffding's inequality to represent uncertainty over a scalar reward
- We would like to now represent uncertainty over r through uncertainty over θ (check your understanding: why is this sufficient to capture uncertainty over r?)
- Requires us to compute an uncertainty set over a vector θ
- This can be done in a computationally tractable way, see e.g. A Contextual-Bandit Approach to Personalized News Article Recommendation, WWW 2010 or Chapter 19 in Lattimore and Szepesvari)

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Generalization and Optimism

- Recall MBIE-EB algorithm for finite state and action domains
- What needs to be modified for continuous / extremely large state and/or action spaces?

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  end while

end loop

Generalization and Optimism

- Recall MBIE-EB algorithm for finite state and action domains
- What needs to be modified for continuous / extremely large state and/or action spaces?
- Estimating uncertainty
  - Counts of (s,a) and (s,a,s') tuples are not useful if we expect only to encounter any state once

Recall: Value Function Approximation with Control

- For Q-learning use a TD target $r + \gamma \max_{a'} \hat{Q}(s', a'; \boldsymbol{w})$ which leverages the max of the current function approximation value

$$\Delta \boldsymbol{w} = \alpha(r(s) + \gamma \max_{a'} \hat{Q}(s', a'; \boldsymbol{w}) - \hat{Q}(s, a; \boldsymbol{w}))\nabla_{\boldsymbol{w}} \hat{Q}(s, a; \boldsymbol{w})$$

- Modify to:

$$\Delta \boldsymbol{w} = \alpha(r(s) + r_{bonus}(s, a) + \gamma \max_{a'} \hat{Q}(s', a'; \boldsymbol{w}) - \hat{Q}(s, a; \boldsymbol{w}))\nabla_{\boldsymbol{w}} \hat{Q}(s, a; \boldsymbol{w})$$

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- $r_{bonus}(s, a)$ should reflect uncertainty about future reward from (s, a)
- Approaches for deep RL that make an estimate of visits / density of visits include: Bellemare et al. NIPS 2016; Ostrovski et al. ICML 2017; Tang et al. NIPS 2017
- Note: bonus terms are computed at time of visit. During episodic replay can become outdated.

Benefits of Strategic Exploration: Montezuma's revenge

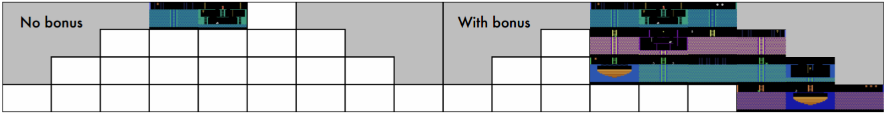


Figure 3: "Known world" of a DQN agent trained for 50 million frames with (**right**) and without (**left**) count-based exploration bonuses, in MONTEZUMA'S REVENGE.

Figure: Bellemare et al. "Unifying Count-Based Exploration and Intrinsic Motivation"

- https://www.youtube.com/watch?v=ToSe_CUG0F4
- Enormously better than standard DQN with ϵ-greedy approach

Generalization and Strategic Exploration: Thompson Sampling

- Leveraging Bayesian perspective has also inspired some approaches
- One approach: Thompson sampling over representation & parameters (Mandel, Liu, Brunskill, Popovic IJCAI 2016)

Generalization and Strategic Exploration: Thompson Sampling

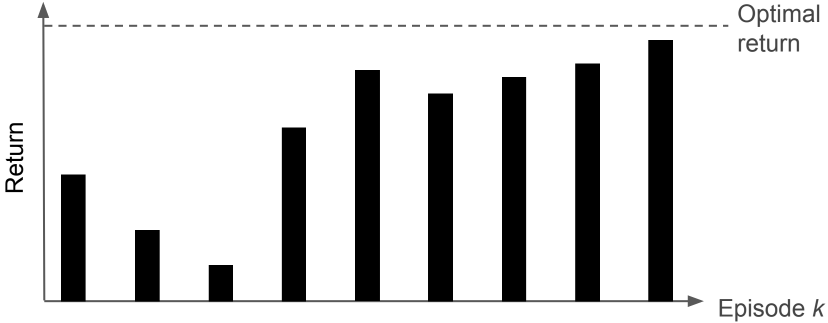
- For scaling up to very large domains, again useful to consider model-free approaches
- Non-trivial: would like to be able to sample from a posterior over possible Q^*
- Bootstrapped DQN (Osband et al. NIPS 2016)
  - Train C DQN agents using bootstrapped samples
  - When acting, choose action with highest Q value over any of the C agents
  - Some performance gain, not as effective as reward bonus approaches

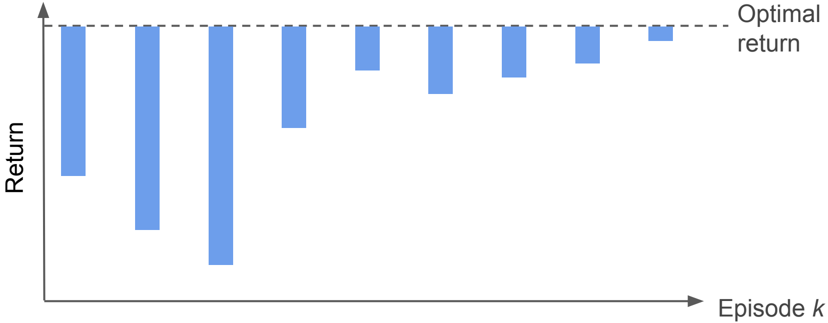
Generalization and Strategic Exploration: Thompson Sampling

- Leveraging Bayesian perspective has also inspired some approaches
- One approach: Thompson sampling over representation & parameters (Mandel, Liu, Brunskill, Popovic IJCAI 2016)
- For scaling up to very large domains, again useful to consider model-free approaches
- Non-trivial: would like to be able to sample from a posterior over possible Q^*
- Bootstrapped DQN (Osband et al. NIPS 2016)
- Efficient Exploration through Bayesian Deep Q-Networks (Azizzadenesheli, Anandkumar, NeurIPS workshop 2017)
  - Use deep neural network
  - On last layer use Bayesian linear regression
  - Be optimistic with respect to the resulting posterior
  - Very simple, empirically much better than just doing linear regression on last layer or bootstrapped DQN, not as good as reward bonuses in some cases

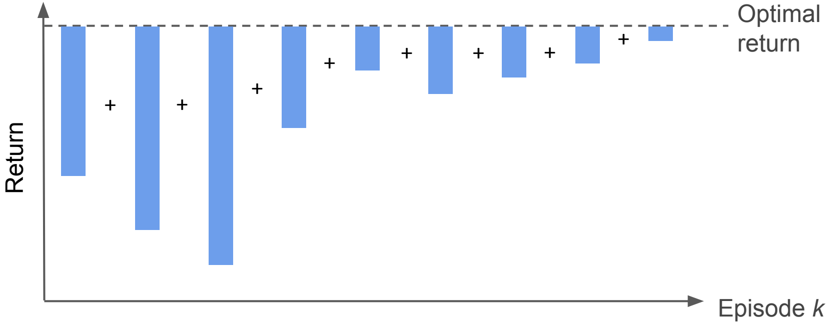
Theoretical Results

Formalisms for Assessing RL Algorithms



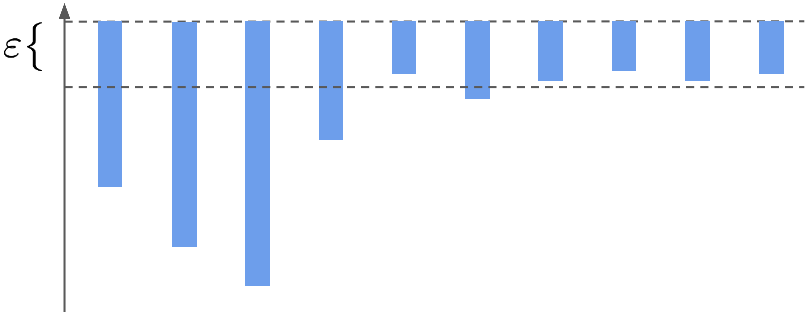

Optimal
return
Return
Episode k

High Probability Regret Bounds for RL



$$P\Big(\underbrace{\sum_t r(s_t, \pi^*(s_t)) - \sum_t r(s_t, \pi_t(s_t))}_{\text{Regret}} \leq F(\delta, \mathcal{S}, \mathcal{A}, T)\Big) \geq 1-\delta$$

May Only Care If Performance Isn't Near Optimal



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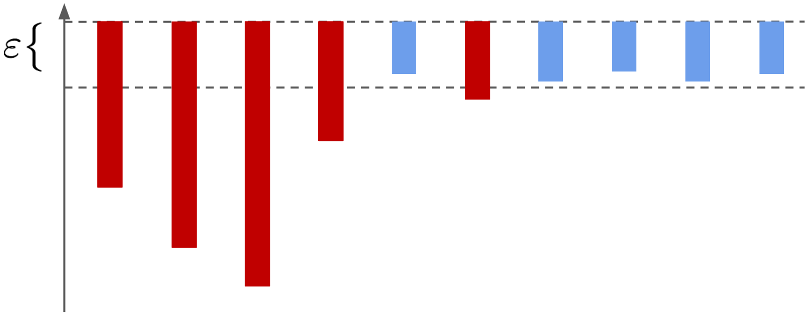
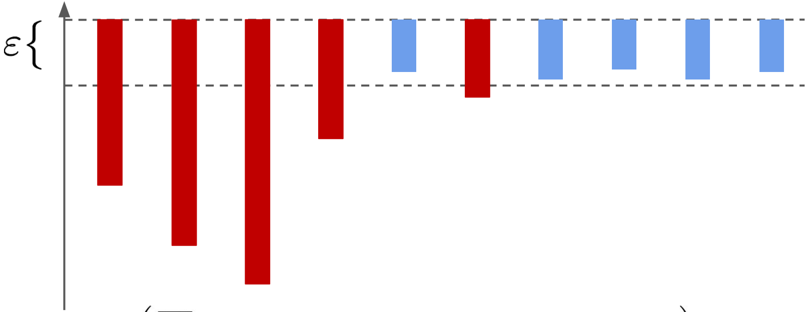
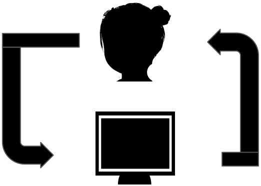


Probably Approximately Correct RL



$$P\left(\underbrace{\sum_t \mathbb{1}\left(V^{\pi_t}(s_t) < V^*(s_t) - \epsilon\right)}_{\text{Sample complexity}} \leq F(\epsilon, \delta, \mathcal{S}, \mathcal{A})\right) \geq 1 - \delta$$

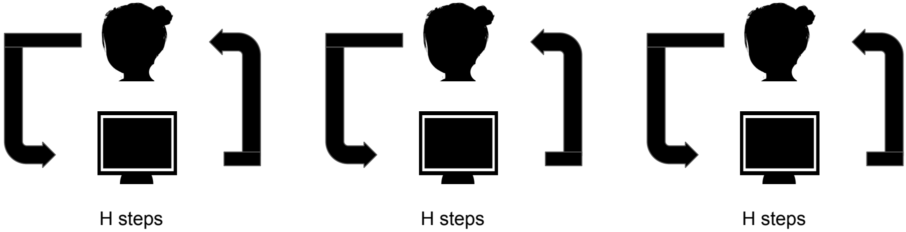
Episodic Tabular Markov Decision Processes



H steps

S: # states
A: # actions
T: # steps
H: time horizon

Episodic Tabular Markov Decision Processes



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A: # actions
T: # steps
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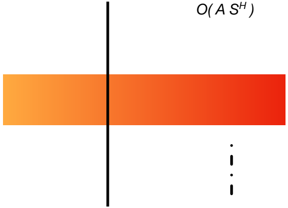
No Intelligent Exploration

O(A S^H)

PAC



Regret

O(T)
(greedy or
epsilon-greedy)

S: # states
A: # actions
T: # steps
H: time horizon

Lower Bound | **Efficient Exploration** | **No Intelligent Exploration**

PAC

- Lower bound: $\tilde{O}\left(\left(\frac{SAH^2}{\epsilon^2} + \frac{S^2AH^3}{\epsilon}\right)\ln\frac{1}{\delta}\right)$ *(Dann, Wei, Li, B. 2019)*
- $\tilde{O}\left(\frac{|\mathcal{S}|^2|\mathcal{A}|H^2}{\epsilon^2}\ln\frac{1}{\delta}\right)$ *(Dann & B 2015)*
- $\tilde{O}\left(\frac{S^2A}{\epsilon^3(1-\gamma)^6}\right)$ *(Kakade 2003; Strehl & Littman 2005)*
- No intelligent exploration: *O(A S^H)*

Regret

- Lower bound: $\tilde{O}(\sqrt{HSAT})$ *(Azar et al. 2017)*
- $\tilde{O}(S\sqrt{HAT})$ *(Dann, Lattimore, B 2017)*
- $\tilde{O}(H\sqrt{SAT})$ *(Dann & B 2015)*
- $\tilde{O}(HS\sqrt{AT})$ *(UCRL2, Jaksch et al. 2010)*
- No intelligent exploration: *O(T) (greedy or epsilon-greedy)*

S: # states
A: # actions
T: # steps
H: time horizon

Lower Bound **Efficient Exploration** **No Intelligent Exploration**

$\tilde{O}\left(\left(\frac{SAH^2}{\epsilon^2}+\frac{S^2AH^3}{\epsilon}\right)\ln\frac{1}{\delta}\right)$.

(Dann, Wei, Li, B. 2019)

(Dann & B 2015)

$\tilde{O}(\frac{|\mathcal{S}|^2|\mathcal{A}|H^2}{\epsilon^2}\ln\frac{1}{\delta})$

(Kakade 2003; Strehl & Littman 2005)

$\tilde{O}\left(\frac{S^2A}{\epsilon^3(1-\gamma)^6}\right)$

O(A S^H)

PAC

Regret

Episodic Tabular RL Closed: Tight upper & lower bounds for episodic tabular RL for both regret & PAC (Dann, Wei, Li, Brunskill ICML 2019)

$\tilde{O}(\sqrt{HSAT})$

(Azar et al. 2017)

$\tilde{O}(S\sqrt{HAT})$

(Dann, Lattimore, B 2017)

(Dann & B 2015)

$\tilde{O}(HS\sqrt{AT})$

(UCRL2, Jaksch et al. 2010)

O(T) (greedy or epsilon-greedy)

S: # states
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Problem Dependent Analysis

Lower Bound

Efficient Exploration

No Intelligent Exploration

PAC

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$\tilde{O}\left(\frac{S^2A}{\epsilon^3(1-\gamma)^6}\right)$

O(A S^H)

Regret

$\tilde{O}(\sqrt{\mathbb{Q}^* SAT})$

(Zanette & B 2019)

$\tilde{O}(\sqrt{HSAT})$

(Azar et al. 2017)

$\tilde{O}(S\sqrt{HAT})$

(Dann, Lattimore, B 2017)

$\tilde{O}(H\sqrt{SAT})$

(Dann & B 2015)

$\tilde{O}(HS\sqrt{AT})$

(UCRL2, Jaksch et al. 2010)

O(T) (greedy or epsilon-greedy)

$\mathbb{Q}^*$: problem dependent constant that does not need to be known

S: # states
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T: # steps
H: time horizon

Problem Dependent Analysis

Lower Bound

Efficient Exploration

No Intelligent Exploration

PAC

$\tilde{O}\left(\left(\frac{SAH^2}{\epsilon^2} + \frac{S^2AH^3}{\epsilon}\right)\ln\frac{1}{\delta}\right).$

(Dann, Wei, Li, B. 2019)

(Dann & B 2015)

$\tilde{O}\left(\frac{|\mathcal{S}|^2|\mathcal{A}|H^2}{\epsilon^2}\ln\frac{1}{\delta}\right)$

(Kakade 2003; Strehl & Littman 2005)

$\tilde{O}\left(\frac{S^2A}{\epsilon^3(1-\gamma)^6}\right)$

O(A S^H)

First Generic Algorithm With Instance Dependent Bounds for Tabular Episodic MDPs (Zanette & Brunskill ICML 2019)

Regret

$\tilde{O}(\sqrt{\mathbb{Q}^* SAT})$

(Zanette & B 2019)

$\tilde{O}(\sqrt{HSAT})$

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$\tilde{O}(HS\sqrt{AT})$

(UCRL2, Jaksch et al. 2010)

O(T) (greedy or epsilon-greedy)

$\mathbb{Q}^*$**: problem dependent constant that does not need to be known**

S: # states
A: # actions
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Early Work: Bound Uncertainty Over Dynamics Model Parameters

$$|Q^\star(s,a) - \widehat{Q}^\star(s,a)| = |p(s,a)^\top V^\star - \widehat{p}(s,a)^\top \widehat{V}^\star|$$

(Assuming no reward error)

$$Q^*(s,a) = r(s,a) + \gamma \sum_{s'} p(s,a) V^*(s')$$

Early Work: Bound Uncertainty Over Dynamics Model Parameters

$$|Q^\star(s,a) - \widehat{Q}^\star(s,a)| = |p(s,a)^\top V^\star - \widehat{p}(s,a)^\top \widehat{V}^\star| \lessapprox \frac{H}{\sqrt{n}}$$

(Hoeffding Inequality)

(Assuming no reward error)

$Q^*(s,a) = r(s,a) + \gamma \sum_{s'} p(s,a) V^*(s')$

Better: Bound Uncertainty *Over Expected Value*

$$|Q^\star(s,a) - \widehat{Q}^\star(s,a)| = |p(s,a)^\top V^\star - \widehat{p}(s,a)^\top \widehat{V}^\star| \lessapprox \frac{H}{\sqrt{n}}$$ *(Hoeffding Inequality)*

(Assuming no reward error)

$$\lessapprox \frac{\sigma_{s,a}^{V^\star}}{\sqrt{n}} + \frac{H}{n}$$ *(Bernstein Inequality)*

$$\sigma_{s,a}^{V^\star} = Var_{s^+ \sim p(s,a)} V^\star$$

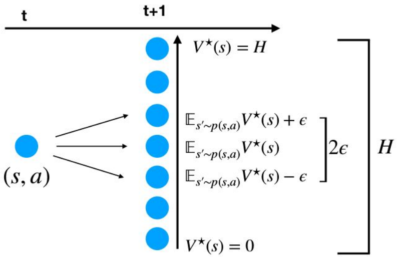
$$Q^*(s,a) = r(s,a) + \gamma \sum_{s'} p(s,a) V^*(s')$$

Better: Bound Uncertainty Over Expected Value
And Use to Create New Optimism Bonuses Used for Decision Making

$$|Q^\star(s,a) - \widehat{Q}^\star(s,a)| = |p(s,a)^\top V^\star - \hat{p}(s,a)^\top \widehat{V}^\star| \lessapprox \frac{H}{\sqrt{n}}$$

(Hoeffding Inequality)

(Assuming no reward error)



$$\lessapprox \frac{\sigma_{s,a}^{V^\star}}{\sqrt{n}} + \frac{H}{n}$$

(Bernstein Inequality)

$$\sigma_{s,a}^{V^\star} = Var_{s+\sim p(s,a)} V^\star$$

$$\mathbb{Q}^* = \max_{s,a} Var_{s'\sim p(s'|s,a)} V^*(s')$$

"Environmental norm" Maillard et al NeurIPS 2014

Problem Dependent Analysis

Lower Bound

Efficient Exploration

No Intelligent Exploration

PAC

$\tilde{O}\left(\left(\frac{SAH^2}{\epsilon^2} + \frac{S^2AH^3}{\epsilon}\right)\ln\frac{1}{\delta}\right)$ *(Dann, Wei, Li, B. 2019)*

$\tilde{O}(\frac{|\mathcal{S}|^2|\mathcal{A}|H^2}{\epsilon^2}\ln\frac{1}{\delta})$ *(Dann & B 2015)*

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O(A S^H)

Regret

$\tilde{O}(\sqrt{\mathbb{Q}^* SAT})$

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$\tilde{O}(HS\sqrt{AT})$ *(UCRL2, Jaksch et al. 2010)*

O(T) (greedy or epsilon-greedy)

$\mathbb{Q}^*$: variance of the value of the next time step

Unlike prior work on instance dependent RL, our algorithm
- does not need as input a problem dependent quantity (vs Bartlett & Tewari 2010; Pazis, Parr & How 2016; Fruit et al, 2018)) and
- matches worst case bounds (vs. Maillard et al. 2014; Talebi et al. 2018; Ortner 2018)

S: # states
A: # actions
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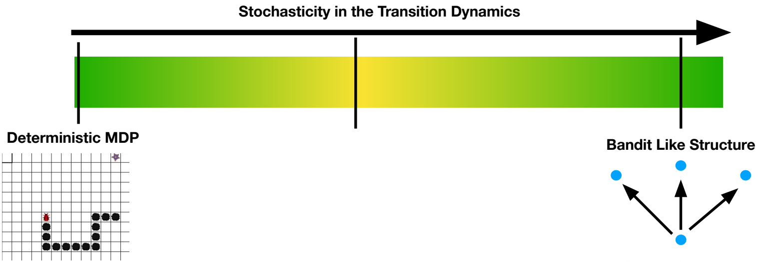
Enhancing Understanding of When it Is Hard to Learn to Act Well

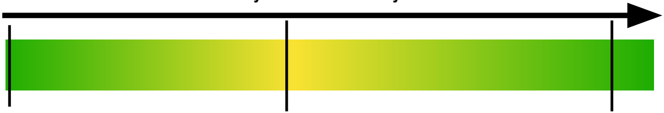


$\tilde{O}(SAH^2)$

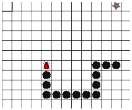
$\tilde{O}(\sqrt{SAT} + [\ldots])$

Enhancing Understanding of When it Is Hard to Learn to Act Well

Stochasticity in the Transition Dynamics

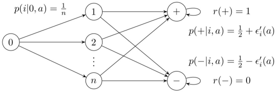


Deterministic MDP



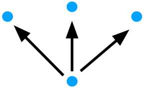
$\tilde{O}(SAH^2)$

Hard Instances
Inducing the Lower Bound



$\tilde{O}(\sqrt{HSAT} + [\dots])$

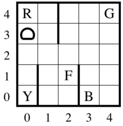
Bandit Like Structure



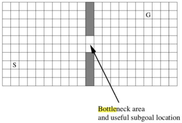
$\tilde{O}(\sqrt{SAT} + [\dots])$

Answers part of COLT open question (by Agarwal & Jiang):
No horizon dependence in regret bound for their setting

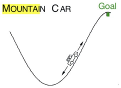
Validates Empirical Findings of Prior Work



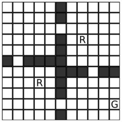
Taxi,
(Dietterich,1998)



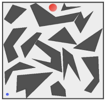
Bottleneck,
(McGovern and Barto, 2001)



Mountain Car,
(Sutton and Barto, 1998)



Red Herring,
(Hester and Stone, 2009)



Pinball,
(Konidaris and Barto, 2009)

$\mathbb{Q}^*$ [the variance of the value of the next state] is numerically small on many common benchmarks: Maillard et al. NeurIPS 2014

Problem Dependent Analysis

Lower Bound

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$O(A\ S^H)$

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$\mathbb{Q}^*$: problem dependent constant that does not need to be known

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Theoretical Results

- Discussed regret bounds for bandits, & PAC bounds for tabular MDPs
- Now exist tight (in dominant term) minimax results for regret and PAC for tabular MDPs
  - Azar, Mohammad Gheshlaghi, Ian Osband, and Rémi Munos. Minimax regret bounds for reinforcement learning. ICML 2017 (regret)
  - Dann, C., Li, L., Wei, W., and Brunskill, E. Policy certificates: Towards accountable reinforcement learning. ICML 2019 (PAC)
- Also exist instance-dependence bounds for tabular MDPs, e.g.:
  - Zanette and Brunskill. Tighter problem-dependent regret bounds in reinforcement learning without domain knowledge using value function bounds. ICML 2019
  - Simchowitz and Jamieson. Non-asymptotic gap-dependent regret bounds for tabular MDPs. NeurIPS 2019.

Theoretical Results: Function Approximation & RL

- Do there exist strong theoretical bounds for RL with function approximation?
- Active area of recent work
  - Jin, Yang, Wang, and Jordan. "Provably efficient reinforcement learning with linear function approximation." COLT 2020.
  - Many others, including our work (lead by Andrea Zanette), and Mengdi Wang's lab.
- Active area: quantifying features of the domain that correspond to hardness
- Eluder dimension (Russo and Van Roy), Bellman rank (Jiang et al), ..

Table of Contents

Summary: What You Are Expected to Know

- Define the tension of exploration and exploitation in RL and why this does not arise in supervised or unsupervised learning
- Be able to define and compare different criteria for "good" performance (empirical, convergence, asymptotic, regret, PAC)
- Be able to map algorithms discussed in detail in class to the performance criteria they satisfy
- Understand the UCB proof sketch
- For those of you doing default project: be able to implement UCB and TS for linear contextual bandit. See e.g. A Contextual-Bandit Approach to Personalized News Article Recommendation, WWW 2010 or Chapter 19 in Lattimore and Szepesvari)

Class Structure

- Last time: Fast Learning (Bayesian bandits to MDPs)
- **This time: Fast Learning III (MDPs)**
- Next time: Batch Offline RL